Topological Phases in Floquet Systems

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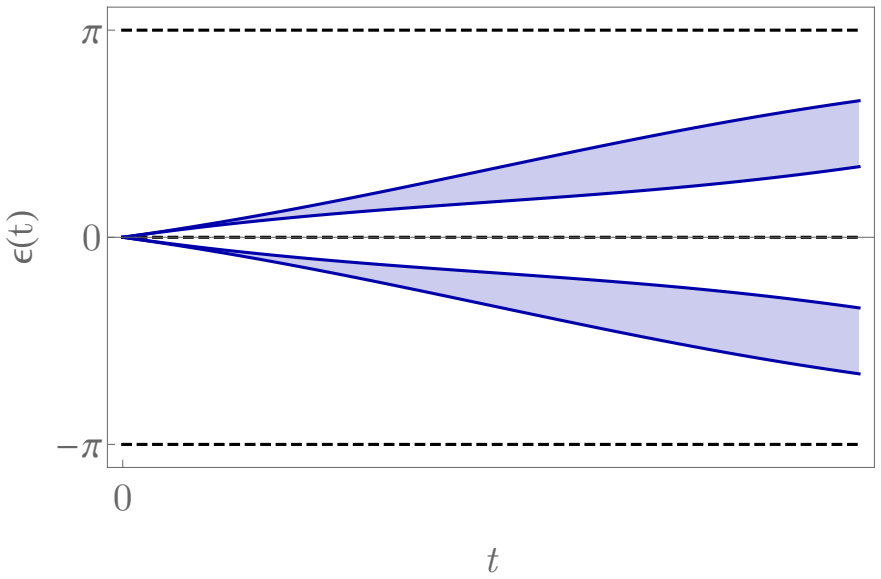
Outline

Topological Insulators and SPT Phases

- Bulk-boundary correspondence...of what?
  - Ground states
  - Hamiltonians
  - Unitaries (for some small time evolution):

$$U(t) = \exp\left(-iHt\right)$$

- 'Quasienergies' $\epsilon_i(t)$ correspond to eigenvalues $e^{i\epsilon_i(t)}$ of $U(t)$.

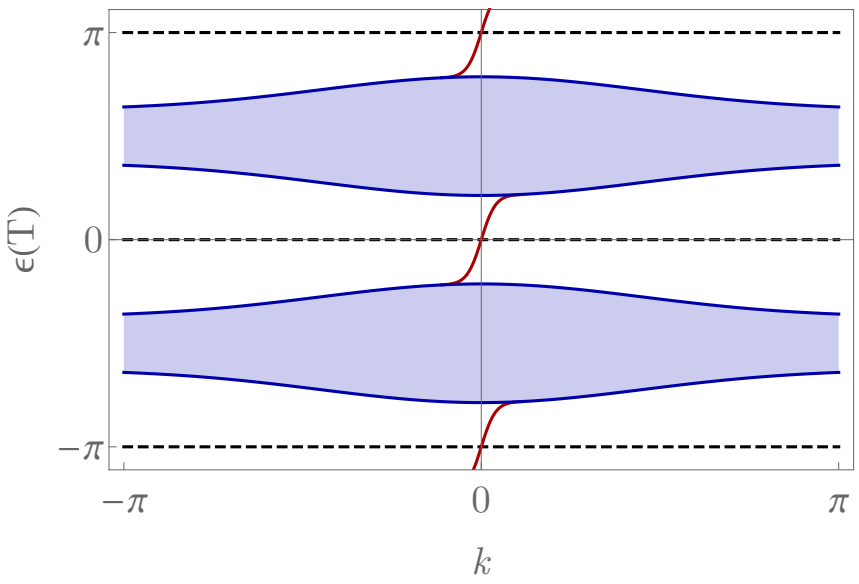


Unitary Evolutions

- What if the Hamiltonian is time-varying?

$$U(t) = \mathcal{T}\left[\exp\left(-i\int_0^t H(t')\mathrm{d}t'\right)\right]$$

- Without loss of generality, we will assume the spectrum contains bulk gaps at $\epsilon = 0$ and $\epsilon = \pi$, when $t = T$.



Floquet Hamiltonians

Question

Given a gapped bulk unitary, are there any robust topological edge states for the different AZ classes that lie outside of the ones that result from a constant H_{top}?

- Consider the Floquet Hamiltonian (with $T = 1$ and branch cut at π)

$$H_F(k) = -i \ln \left[U(k, 1) \right].$$

- Let $U(t)$ have eigenvalues $e^{i\epsilon(t)}$ and result from evolution due to

$$H(t) = H_0 + H'(t),$$

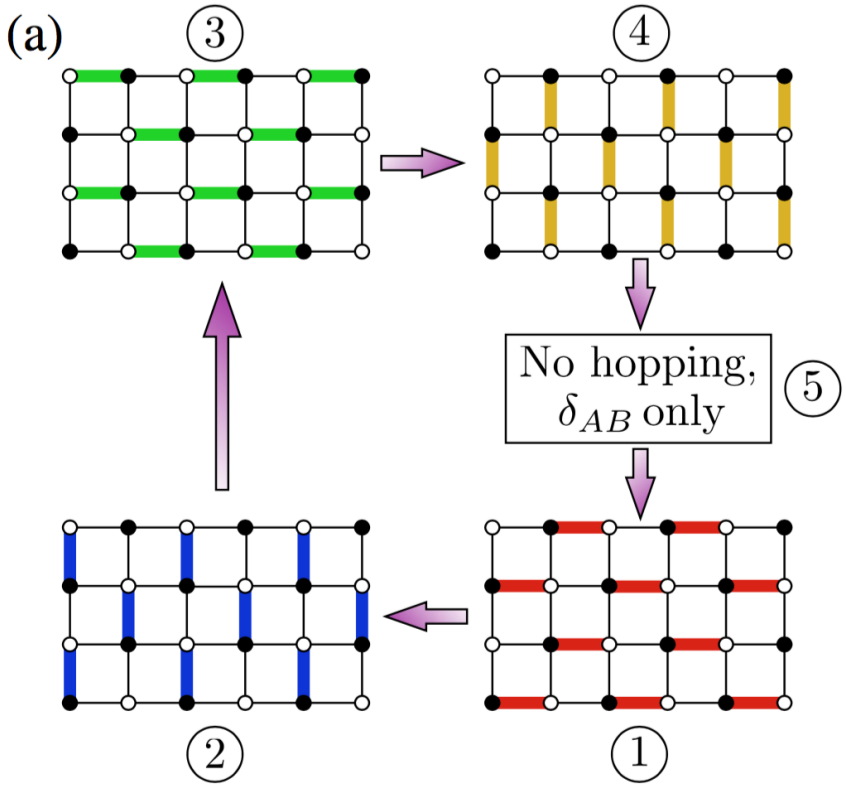
  with H_0 topologically trivial.
- If H_F is topologically non-trivial, then we expect edge modes.

Rudner System

Question

Given a gapped bulk unitary, are there any robust edge states that can't be accounted for by a non-trivial H_F?

Consider the five-part drive of Rudner et al. [PRX 3, 031005 (2013)]

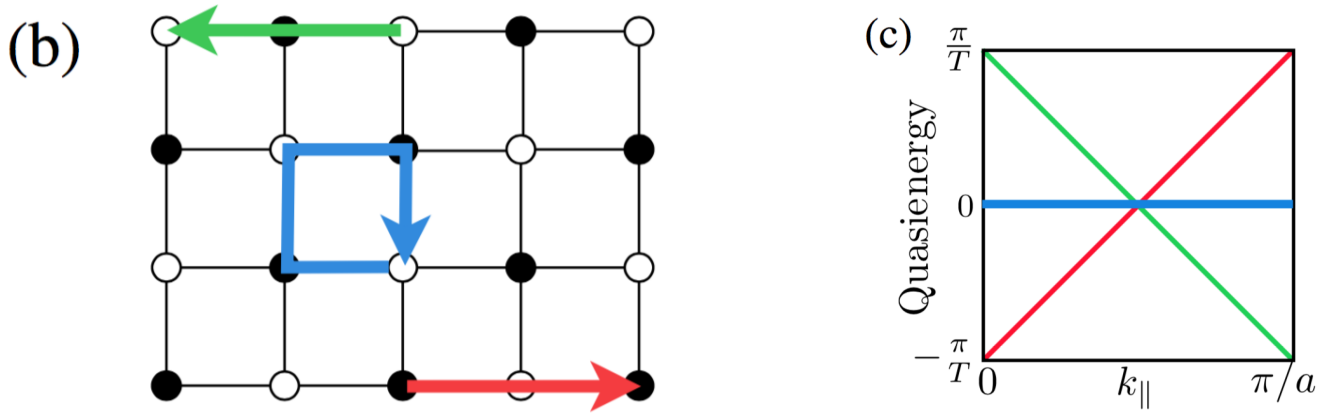


1. A $\rightarrow$
2. A $\uparrow$
3. A $\leftarrow$
4. A $\downarrow$
5. Apply on-site energy

Rudner System

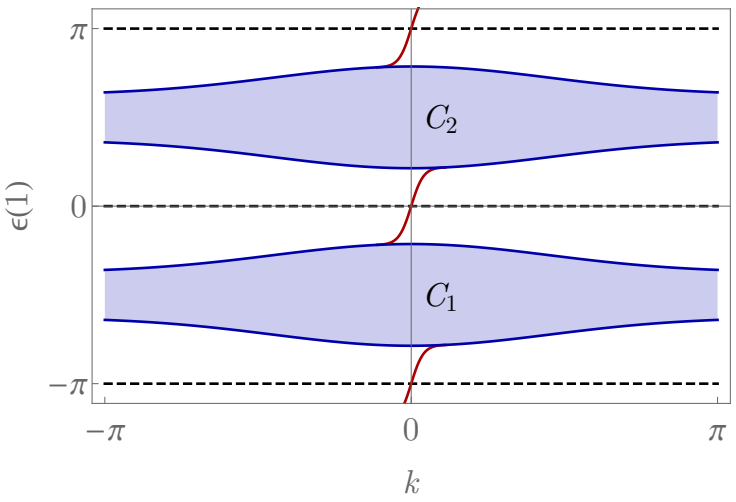
After one complete cycle:

- Bulk particles return to their original position.
- Particles on B-sites on top edge move two units to the left.
- Particles on A-sites on bottom edge move two units to the right.



$$U_{\text{bulk}}(1) = \mathbb{I} \qquad \text{BUT} \qquad U_{\text{edge}}(1) \neq \mathbb{I}$$

Topological Invariants



The number of edge modes in a gap is:

$$N_{\text{edge}} = C_+ - C_- + N_U \equiv W[U]$$

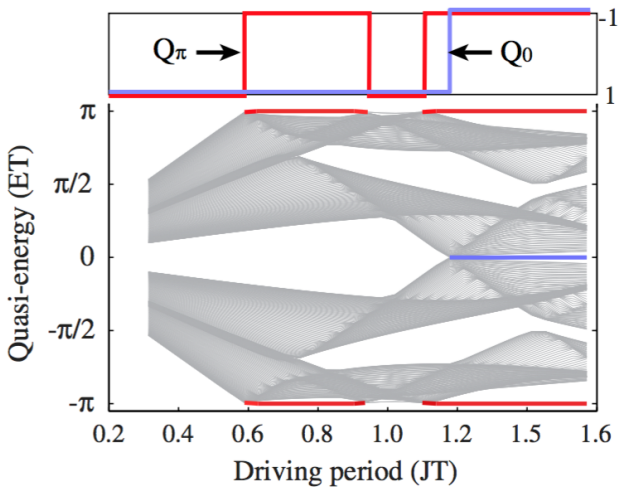
- The winding number is defined by

$$W[U] = \frac{1}{8\pi^2} \int \mathrm{d}t \mathrm{d}k_x \mathrm{d}k_y \mathrm{Tr}\left(U^{-1}\partial_t U \left[U^{-1}\partial_{k_x} U, U^{-1}\partial_{k_y} U\right]\right),$$

where U is an associated unitary cycle defined on a 3-torus.

Other Classes

- Class D in $d = 1$:
  - Jiang et al. [PRL 106, 220402 (2011)]
  - Thakurathi et al. [PRB 88, 155133 (2013)]



Question

Can we extend these results to other symmetry classes and dimensions, analogous to Kitaev's periodic table of topological insulators and superconductors [Kitaev 2009]?

Outline

Classification of Unitaries

Theorem

If G is the static topological insulator classification, then the classification of Floquet topological insulators for a unitary with gaps at 0 and π is $G \times G$.

Some elements of the periodic table had been filled earlier, notably:

- Class D in $d = 1$.
- Class AIII in $d = 2, 3$.
- Class A in $d = 2$.

Most elements were missing, however, and there were some errors (for instance in classifying class AIII in various dimensions).

Symmetries for Unitary Evolution

For any AZ symmetry class $\mathcal{S}$, we:

1. Require that H_F have the defining symmetries of Hamiltonians in $\mathcal{S}$.
2. Require the instantaneous Hamiltonian $H(t)$ to belong to the AZ symmetry class $\mathcal{S}$.

The 10 AZ classes are based on:

- Particle-hole symmetry,

$$\mathcal{P} = \mathcal{K}P \quad \Rightarrow \quad PHP^{-1} = -H^*$$

- Time-reversal symmetry,

$$\Theta = \mathcal{K}\theta \quad \Rightarrow \quad \theta H \theta^{-1} = +H^*$$

- Chiral symmetry,

$$C \quad \Rightarrow \quad CHC^{-1} = -H$$

Symmetries for Unitary Evolution

- For PHS, we can satisfy both symmetry requirements 1 and 2.
- For TRS and Chiral symmetry, requirements 1 and 2 are incompatible: a generic $H(t)$ satisfying requirement 2 does not satisfy requirement 1.
- For TRS, a natural time-dependent symmetry definition is

$$\theta H(t^* + t)\theta^{-1} = H^*(t^* - t),$$

leading to a Floquet Hamiltonian H_F that satisfies

$$\theta H_F \theta^{-1} = H_F^*.$$

- For chiral symmetry, there is no natural time-dependent symmetry condition. However, the choice

$$CH(t^* + t)C^{-1} = -H(t^* - t),$$

lends itself to a topological classification of unitaries.

Composition of Unitaries

For $U_1, U_2 \in$ AZ class $\mathcal{S}$, we define $U_1 * U_2$ as the sequential evolution of each unitary in turn:

- If $\mathcal{S}$ has no TRS and is not chiral, then

$$H(t) = \begin{cases} H_1(\mathbf{k}, 2t) & 0 \leq t \leq 1/2 \\ H_2(\mathbf{k}, 2t-1) & 1/2 \leq t \leq 1 \end{cases} .$$

- Otherwise, we define the composition through

$$H(t) = \begin{cases} H_2(\mathbf{k}, 2t) & 0 \leq t \leq 1/4 \\ H_1(\mathbf{k}, 2t-1/2) & 1/4 \leq t \leq 3/4 \\ H_2(\mathbf{k}, 2t-1) & 3/4 \leq t \leq 1 \end{cases} ,$$

which ensures that $U_1 * U_2 \in \mathcal{S}$.

Equivalence Classes of Unitaries

- Denote space of gapped unitaries within the AZ symmetry class $\mathcal{S}$ as $\mathcal{U}_g^{\mathcal{S}}$
- We write $U_1 \approx U_2$ if U_1 is homotopic to U_2 within $\mathcal{U}_g^{\mathcal{S}}$
- We define $U_1 \sim U_2 \iff \exists$ trivial evolutions $U_{n_1}^0$ and $U_{n_2}^0$, such that

$$U_1 \oplus U_{n_1}^0 \quad \approx \quad U_2 \oplus U_{n_2}^0,$$

- Finally, for pairs (U_1, U_2) and (U_3, U_4), we write

$$(U_1, U_2) \sim (U_3, U_4) \iff U_1 \oplus U_4 \quad \sim \quad U_2 \oplus U_3.$$

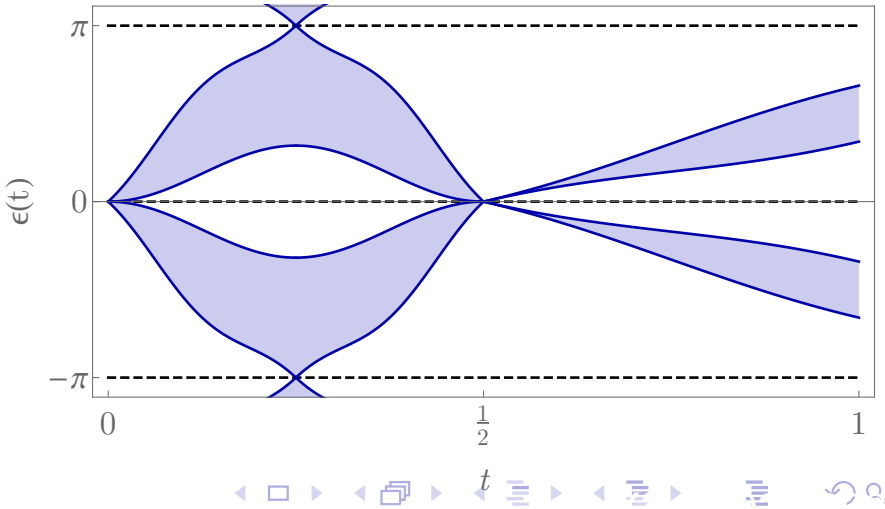
Effective Decomposition of Unitaries

- Defn: Unitary loops : $U(\mathbf{k}, 0) = U(\mathbf{k}, 1) = \mathbb{I}$.
- Defn: Const H Evolution: $U(\mathbf{k}, t) = e^{-iH(\mathbf{k})t}$ for some $H(\mathbf{k})$, whose eigenvalues have magnitude strictly less than π.

Theorem

Every unitary $U \in \mathcal{U}^{\mathcal{S}}_{0,\pi}$ can be continuously deformed to a composition of a unitary loop L and a constant Hamiltonian evolution C, which we write as $U \approx L * C$. L and C are unique up to homotopy.

- We can classify *any* gapped unitary by separately classifying the loop component and the constant Hamiltonian evolution component.



Classification of Constant H Unitaries

- Label the set of constant Hamiltonian evolutions in symmetry class $\mathcal{S}$ as $\mathcal{U}_C^{\mathcal{S}}$.
- Label the set of static gapped Hamiltonians in symmetry class $\mathcal{S}$, whose eigenvalues E satisfy $0 < |E| < \pi$, by $\mathcal{H}^{\mathcal{S}}$.
- The set of gapped constant evolutions in $\mathcal{U}_C^{\mathcal{S}}$ is clearly in one-to-one correspondence with the set of static Hamiltonians in $\mathcal{H}^{\mathcal{S}}$.

$\Rightarrow$ Constant Hamiltonian evolutions are classified according to the static topological insulator periodic table [Kitaev 2009]

K Theory Classification

To classify the loop unitary components, we construct a group as follows:

- Take pairs (U_1, U_2) and consider the operation '$+$' defined through

$$(U_1, U_2) + (U_3, U_4) \quad = \quad (U_1 \oplus U_3, U_2 \oplus U_4),$$

  where $\oplus$ is the direct sum

- We map the problem onto the more standard problem of finding the group of equivalence classes of Hermitian maps H_U on $S^1 \times X$ with a set of symmetries $\mathcal{S}'$:

$$U(\mathbf{k}, t) \quad \leftrightarrow \quad H_U(\mathbf{k}, t) = \begin{pmatrix} 0 & U(\mathbf{k}, t) \\ U^\dagger(\mathbf{k}, t) & 0 \end{pmatrix}$$

  [$\mathcal{S}'$ is a set of symmetry operators derived from $\mathcal{S}$; X is the Brillouin zone].

- Using Morita equivalence, we reduce this to a standard K group.

K Theory Classification

For each symmetry class, we find:

$$
\begin{array}{lcll}
K^1(S^1 \times X, \{0\} \times X) & = & K^0(X) & \text{Class A} \\
K^2(S^1 \times X, \{0\} \times X) & = & K^1(X) & \text{Class AIII}
\end{array}
$$

$$
\begin{array}{lcll}
KR^{0,7}(S^1 \times X, \{0\} \times X) & = & KR^{0,0}(X) & \text{Class AI} \\
KR^{0,0}(S^1 \times X, \{0\} \times X) & = & KR^{0,1}(X) & \text{Class BDI} \\
KR^{0,1}(S^1 \times X, \{0\} \times X) & = & KR^{0,2}(X) & \text{Class D} \\
KR^{0,2}(S^1 \times X, \{0\} \times X) & = & KR^{0,3}(X) & \text{Class DIII} \\
KR^{0,3}(S^1 \times X, \{0\} \times X) & = & KR^{0,4}(X) & \text{Class AII} \\
KR^{0,4}(S^1 \times X, \{0\} \times X) & = & KR^{0,5}(X) & \text{Class CII} \\
KR^{0,6}(S^1 \times X, \{0\} \times X) & = & KR^{0,7}(X) & \text{Class CI} \\
KR^{0,5}(S^1 \times X, \{0\} \times X) & = & KR^{0,6}(X) & \text{Class C}
\end{array}
$$

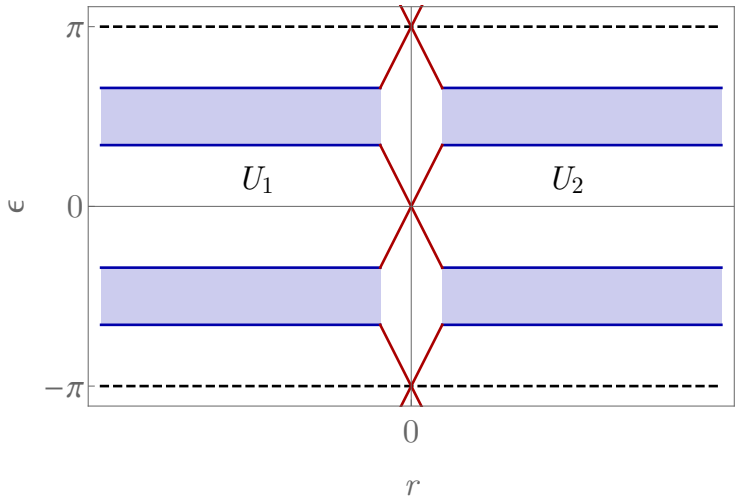
Complete Classification of Unitaries

- Combining the loop and constant evolution components, two-gapped unitaries are classified according to the following table:

$\mathcal{S}$	$d=0$	1	2	3	4	5	6	7
A	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$
AIII	$\emptyset$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\mathbb{Z}\times\mathbb{Z}$
AI	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
BDI	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\mathbb{Z}_2\times\mathbb{Z}_2$
D	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$
DIII	$\emptyset$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathbb{Z}\times\mathbb{Z}$
AII	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\emptyset$	$\emptyset$
CII	$\emptyset$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\emptyset$
C	$\emptyset$	$\emptyset$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$
CI	$\emptyset$	$\emptyset$	$\emptyset$	$\mathbb{Z}\times\mathbb{Z}$	$\emptyset$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}$

Bulk-Edge Correspondence

- At an interface between two systems, the bulk topology may be manifested as edge modes.



- If n_C is the constant evolution invariant and n_L is the unitary loop invariant, then the number of edge modes in each gap is:

$$\begin{aligned} n_\pi &= n_L \\ n_0 &= n_C + n_L \end{aligned}$$

Outline

Class D Hamiltonians

- General Class D Hamiltonian

$$H = \frac{i}{4}\sum_{ij}\gamma_i A_{ij}\gamma_j,$$

where $\gamma_i^\dagger = \gamma_i$ are Majorana fermions and A_{ij} is an antisymmetric matrix.

- In d=1, the static classification is $\mathbb{Z}_2$ [Kitaev 2001].

Class D Unitaries in d=1

- For a time dependent Hamiltonian, let

$$O(t) = \mathcal{T} \exp\left(\int_0^t A_{ij}(t')\mathrm{d}t'\right).$$

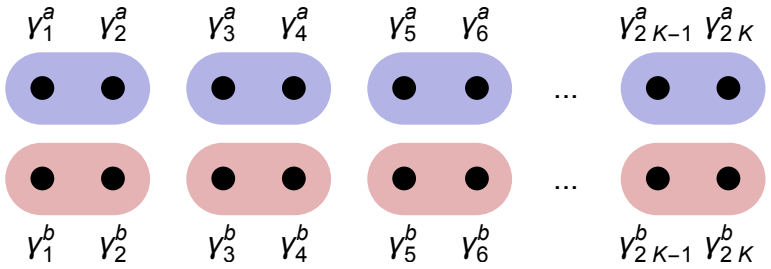
- $O(t)$ belongs to the special orthogonal group, $SO(2n)$.
- $H(0)$ and $H(\pi)$ have the full symmetry of a 0d Class D Hamiltonian.

$\implies$ 1d Floquet cycles in Class D can be characterized by an invariant $\in \mathbb{Z}_2 \times \mathbb{Z}_2$ for $n > 2$.

- This agrees with the classification obtained using K-theory

Non-trivial Unitary Loop

- Consider a one-dimensional fermionic chain of length K that has a two-state Hilbert space at each site (with annihilation operators a_j and b_j):



- We define two sets of Majorana operators through

$$\gamma_{2j-1}^a = a_j + a_j^\dagger, \qquad \gamma_{2j}^a = \frac{a_j - a_j^\dagger}{i}$$

$$\gamma_{2j-1}^b = b_j + b_j^\dagger, \qquad \gamma_{2j}^b = \frac{b_j - b_j^\dagger}{i},$$

which satisfy $\gamma^\dagger = \gamma$.

Non-trivial Unitary Loop

- Let

$$
\begin{aligned}
H_{1a} &= -\sum_j \left(a_j^\dagger a_j - \frac{1}{2} \right) = -\frac{i}{2} \sum_j \left(\gamma_{2j-1}^a \gamma_{2j}^a \right) \\
H_{2a} &= \frac{1}{2} \sum_j \left(-a_j^\dagger a_{j+1} - a_{j+1}^\dagger a_j + a_j a_{j+1} + a_{j+1}^\dagger a_j^\dagger \right) \\
&= \frac{i}{2} \sum_j \left(\gamma_{2j}^a \gamma_{2j+1}^a \right),
\end{aligned}
$$

  with H_{1b}, H_{2b} defined identically in terms of the b operators and γ^b Majoranas.

- H_{1a} and H_{2a} respectively correspond to the trivial and nontrivial phases of the 1D class D superconductor.

Non-trivial Unitary Loop

- Evolve the system with $H_1 = H_{1a} + 2H_{1b}$ for $0 \leq t \leq \pi$ and $H_2 = H_{2a} + 2H_{2b}$ for $\pi \leq t \leq 2\pi$.
- H_{2a} and H_{2b} are topologically non-trivial while H_{1a}, H_{1b} are trivial.
- The evolution by H_{1a} pushes the Majorana mode of subsystem a to quasienergy $\epsilon = \pi$,
- Evolving the closed system until $t = 2\pi$ leads to a unitary that is the identity (up to an overall phase factor).

Unitaries for Open and Closed Systems

- Terms $\frac{i}{2}\gamma^a_{2K}\gamma^a_1$ from H_{2a} and $\frac{i}{2}\gamma^b_{2K}\gamma^b_1$ from H_{2b} are absent in the open system Hamiltonians.
- With

$$d = \frac{1}{2}\left(\gamma^a_{2K} + i\gamma^a_1\right), \qquad d^\dagger = \frac{1}{2}\left(\gamma^a_{2K} - i\gamma^a_1\right),$$
$$e = \frac{1}{2}\left(\gamma^b_{2K} + i\gamma^b_1\right), \qquad e^\dagger = \frac{1}{2}\left(\gamma^b_{2K} - i\gamma^b_1\right),$$

the unitary of the open system may be written

$$\begin{aligned} U_{\rm op}(2\pi) &= e^{\left[\frac{\pi}{2}\left(\gamma^a_{2K}\gamma^a_1 + 2\gamma^b_{2K}\gamma^b_1\right)\right]} U_{\rm cl}(2\pi) \\ &= e^{\left[-\frac{i\pi}{2}\left(d^\dagger d - dd^\dagger\right) - i\pi\left(e^\dagger e - ee^\dagger\right)\right]} U_{\rm cl}(2\pi). \end{aligned}$$

Effective Unitary for the Edge

- Effective Unitary for the edge:

$$U_{\text{eff}}(2\pi) \quad = \quad e^{\left[-\frac{i\pi}{2}\left(d^\dagger d - dd^\dagger\right) - i\pi\left(e^\dagger e - ee^\dagger\right)\right]},$$

- Define effective parity operator $\hat{P}_L$ at the left edge of the open chain
- Then,

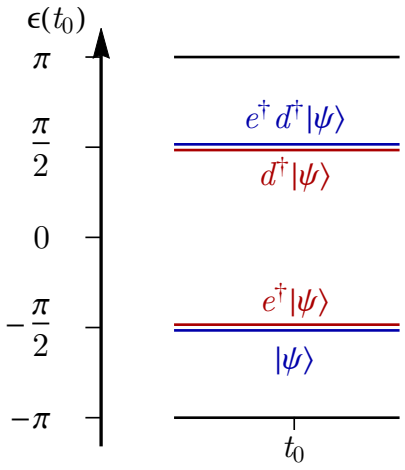
$$\{\hat{P}_L, U_{\text{eff}}\} \quad = \quad 0.$$

Many body picture

- Bulk topological invariant: $U_0(T)U_\pi^\dagger(T) = e^{i\nu\pi}\mathbb{I}$
  - correctly predicts π Majorana mode
  - agrees with the non-interacting invariant
- Edge picture:
  - two fold degeneracy in spectrum of open system
  - has effective unitary of the form described above

Phase Transitions

- Suppose there is phase transition at t_0.
  i.e., no edge π Majorana modes for $t < t_0$ but Majorana modes exist for $t > t_0$.
- At t_0, each bulk eigenstate $|\psi\rangle$ becomes part of a multiplet $\{|\psi\rangle, d^\dagger|\psi\rangle, e^\dagger|\psi\rangle, d^\dagger e^\dagger|\psi\rangle\}$ after the transition.
- These form two pairs separated by quasienergy π (modulo 2π). The two states in each pair have opposite parity.



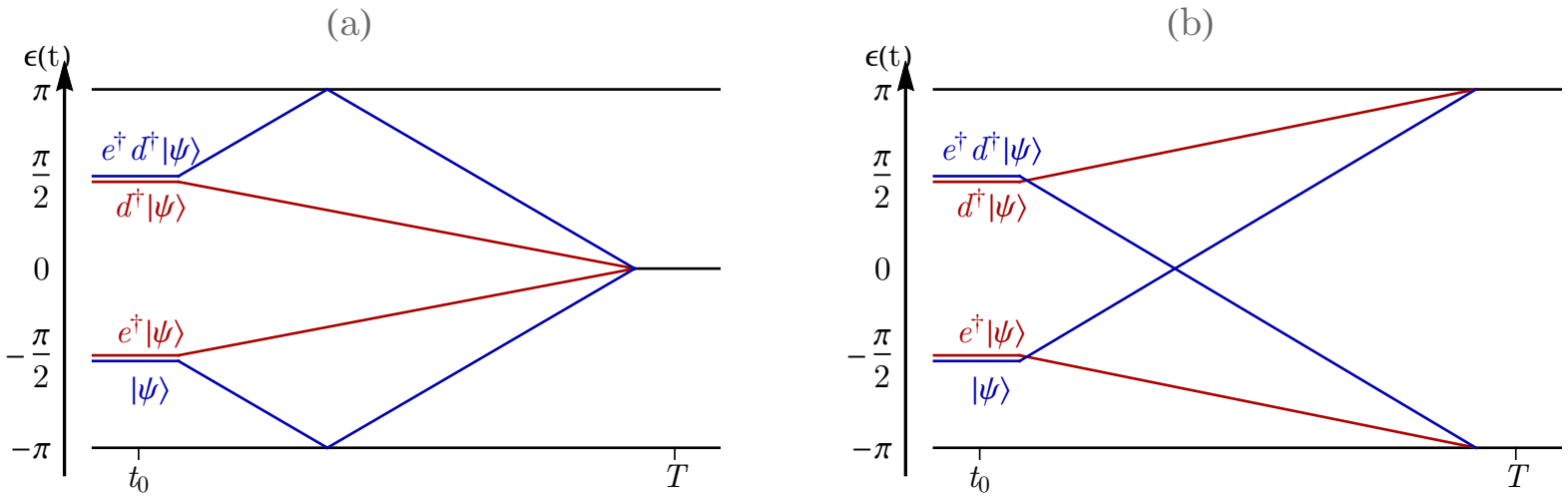
Bulk Boundary correspondence

- Suppose at some point beyond t_0, we reconnect the edges with an edge Hamiltonian

$$H' = h_d \left(dd^\dagger - d^\dagger d \right) + h_e \left(ee^\dagger - e^\dagger e \right),$$

and choose h_d and h_e so that the unitary at the end of the evolution is $\mathbb{I}$

- If we reconnect with a π flux, h_d, h_e change sign, the unitary goes to $-\mathbb{I}$.



Conclusions

- Topological Floquet systems may be classified using K theory.
- A gapped unitary evolution can be uniquely decomposed (up to homotopy) into a loop component and a constant evolution.
- The periodic table for free fermionic unitary evolutions may be obtained from the static periodic table through $G \to G \times G$.

- Unitary loops in one-dimensional interacting systems in class D may be classified by considering the effective edge unitary.
- A topological invariant is given by the phase change under a flux insertion.