Topological Phases in Floquet Systems

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Outline

1. Introduction

2. Free Fermionic Systems

3. Interacting Systems in Class D
Topological Insulators and SPT Phases

- Bulk-boundary correspondence...of what?
  - Ground states
  - Hamiltonians
  - Unitaries (for some small time evolution):
    \[ U(t) = \exp(-iHt) \]

- ‘Quasienergies’ \( \epsilon_i(t) \) correspond to eigenvalues \( e^{i\epsilon_i(t)} \) of \( U(t) \).
What if the Hamiltonian is time-varying?

\[ U(t) = \mathcal{T} \exp \left( -i \int_0^t H(t') dt' \right) \]

Without loss of generality, we will assume the spectrum contains bulk gaps at \( \epsilon = 0 \) and \( \epsilon = \pi \), when \( t = T \).
Question

Given a gapped bulk unitary, are there any robust topological edge states for the different AZ classes that lie outside of the ones that result from a constant $H_{\text{top}}$?

- Consider the Floquet Hamiltonian (with $T = 1$ and branch cut at $\pi$)
  
  $$H_F(k) = -i \ln [U(k, 1)].$$

- Let $U(t)$ have eigenvalues $e^{i\epsilon(t)}$ and result from evolution due to
  
  $$H(t) = H_0 + H'(t),$$
  
  with $H_0$ topologically trivial.

- If $H_F$ is topologically non-trivial, then we expect edge modes.
Question

Given a gapped bulk unitary, are there any robust edge states that can’t be accounted for by a non-trivial $H_F$?

Consider the five-part drive of Rudner et al. [PRX 3, 031005 (2013)]

(a)

1. A $\rightarrow$
2. A $\uparrow$
3. A $\leftarrow$
4. A $\downarrow$
5. Apply on-site energy
After one complete cycle:

- Bulk particles return to their original position.
- Particles on B-sites on top edge move two units to the left.
- Particles on A-sites on bottom edge move two units to the right.

\[ U_{\text{bulk}}(1) = \mathbb{I} \quad \text{BUT} \quad U_{\text{edge}}(1) \neq \mathbb{I} \]
The number of edge modes in a gap is:

\[ N_{\text{edge}} = C_+ - C_- + N_U \equiv W[U] \]

- The winding number is defined by

\[
W[U] = \frac{1}{8\pi^2} \int dt dk_x dk_y \text{Tr} \left( U^{-1} \partial_t U \left[ U^{-1} \partial_{k_x} U, U^{-1} \partial_{k_y} U \right] \right),
\]

where U is an associated unitary cycle defined on a 3-torus.
Other Classes

- Class D in $d = 1$:
  - Jiang et al. [PRL 106, 220402 (2011)]
  - Thakurathi et al. [PRB 88, 155133 (2013)]

**Question**

Can we extend these results to other symmetry classes and dimensions, analogous to Kitaev’s periodic table of topological insulators and superconductors [Kitaev 2009]?
Outline

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Classification of Unitaries

Theorem

If $G$ is the static topological insulator classification, then the classification of Floquet topological insulators for a unitary with gaps at $0$ and $\pi$ is $G \times G$.

Some elements of the periodic table had been filled earlier, notably:

- Class D in $d = 1$.
- Class AIII in $d = 2, 3$.
- Class A in $d = 2$.

Most elements were missing, however, and there were some errors (for instance in classifying class AIII in various dimensions).
For any AZ symmetry class $S$, we:

1. Require that $H_F$ have the defining symmetries of Hamiltonians in $S$.
2. Require the instantaneous Hamiltonian $H(t)$ to belong to the AZ symmetry class $S$.

The 10 AZ classes are based on:

- **Particle-hole symmetry,**
  
  \[
  \mathcal{P} = \mathcal{K}P \implies PHP^{-1} = -H^* 
  \]

- **Time-reversal symmetry,**
  
  \[
  \Theta = \mathcal{K}\theta \implies \theta H\theta^{-1} = +H^* 
  \]

- **Chiral symmetry,**
  
  \[
  C \implies CHC^{-1} = -H 
  \]
Symmetries for Unitary Evolution

- For PHS, we can satisfy both symmetry requirements 1 and 2.
- For TRS and Chiral symmetry, requirements 1 and 2 are incompatible: a generic $H(t)$ satisfying requirement 2 does not satisfy requirement 1.
- For TRS, a natural time-dependent symmetry definition is

$$\theta H(t^* + t)\theta^{-1} = H^*(t^* - t),$$

leading to a Floquet Hamiltonian $H_F$ that satisfies

$$\theta H_F \theta^{-1} = H_F^*.$$

- For chiral symmetry, there is no natural time-dependent symmetry condition. However, the choice

$$CH(t^* + t)C^{-1} = -H(t^* - t),$$

lends itself to a topological classification of unitaries.
For $U_1, U_2 \in AZ$ class $S$, we define $U_1 \ast U_2$ as the sequential evolution of each unitary in turn:

- If $S$ has no TRS and is not chiral, then
  \[
  H(t) = \begin{cases} 
  H_1(k, 2t) & 0 \leq t \leq 1/2 \\
  H_2(k, 2t - 1) & 1/2 \leq t \leq 1
  \end{cases}.
  \]

- Otherwise, we define the composition through
  \[
  H(t) = \begin{cases} 
  H_2(k, 2t) & 0 \leq t \leq 1/4 \\
  H_1(k, 2t - 1/2) & 1/4 \leq t \leq 3/4 \\
  H_2(k, 2t - 1) & 3/4 \leq t \leq 1
  \end{cases},
  \]

which ensures that $U_1 \ast U_2 \in S$. 
Denote space of gapped unitaries within the AZ symmetry class $S$ as $U_g^S$.

We write $U_1 \approx U_2$ if $U_1$ is homotopic to $U_2$ within $U_g^S$.

We define $U_1 \sim U_2 \iff \exists$ trivial evolutions $U^0_{n_1}$ and $U^0_{n_2}$, such that

$$U_1 \oplus U^0_{n_1} \approx U_2 \oplus U^0_{n_2},$$

Finally, for pairs $(U_1, U_2)$ and $(U_3, U_4)$, we write

$$(U_1, U_2) \sim (U_3, U_4) \iff U_1 \oplus U_4 \sim U_2 \oplus U_3.$$
Effective Decomposition of Unitaries

- **Defn**: Unitary loops: \( U(k, 0) = U(k, 1) = I \).
- **Defn**: Const H Evolution: \( U(k, t) = e^{-iH(k)t} \) for some \( H(k) \), whose eigenvalues have magnitude strictly less than \( \pi \).

**Theorem**

*Every unitary \( U \in \mathcal{U}_{0,\pi}^S \) can be continuously deformed to a composition of a unitary loop \( L \) and a constant Hamiltonian evolution \( C \), which we write as \( U \approx L \ast C \). \( L \) and \( C \) are unique up to homotopy.*

- We can classify *any* gapped unitary by separately classifying the loop component and the constant Hamiltonian evolution component.
Classification of Constant H Unitaries

- Label the set of constant Hamiltonian evolutions in symmetry class $S$ as $\mathcal{U}_C^S$.
- Label the set of static gapped Hamiltonians in symmetry class $S$, whose eigenvalues $E$ satisfy $0 < |E| < \pi$, by $\mathcal{H}^S$.
- The set of gapped constant evolutions in $\mathcal{U}_C^S$ is clearly in one-to-one correspondence with the set of static Hamiltonians in $\mathcal{H}^S$.

$\Rightarrow$ Constant Hamiltonian evolutions are classified according to the static topological insulator periodic table [Kitaev 2009]
To classify the loop unitary components, we construct a group as follows:

- Take pairs \((U_1, U_2)\) and consider the operation ‘+’ defined through

\[
(U_1, U_2) + (U_3, U_4) = (U_1 \oplus U_3, U_2 \oplus U_4),
\]

where \(\oplus\) is the direct sum.

- We map the problem onto the more standard problem of finding the group of equivalence classes of Hermitian maps \(H_U\) on \(S^1 \times X\) with a set of symmetries \(S'\):

\[
U(k, t) \leftrightarrow H_U(k, t) = \begin{pmatrix} 0 & U(k, t) \\ U^\dagger(k, t) & 0 \end{pmatrix}
\]

\([S'\) is a set of symmetry operators derived from \(S\); \(X\) is the Brillouin zone].

- Using Morita equivalence, we reduce this to a standard K group.
For each symmetry class, we find:

\[
\begin{align*}
K^1(\mathbb{S}^1 \times X, \{0\} \times X) &= K^0(X) & \text{Class A} \\
K^2(\mathbb{S}^1 \times X, \{0\} \times X) &= K^1(X) & \text{Class AIII} \\
KR^{0,7}(\mathbb{S}^1 \times X, \{0\} \times X) &= KR^{0,0}(X) & \text{Class Al} \\
KR^{0,0}(\mathbb{S}^1 \times X, \{0\} \times X) &= KR^{0,1}(X) & \text{Class BDI} \\
KR^{0,1}(\mathbb{S}^1 \times X, \{0\} \times X) &= KR^{0,2}(X) & \text{Class D} \\
KR^{0,2}(\mathbb{S}^1 \times X, \{0\} \times X) &= KR^{0,3}(X) & \text{Class DIII} \\
KR^{0,3}(\mathbb{S}^1 \times X, \{0\} \times X) &= KR^{0,4}(X) & \text{Class All} \\
KR^{0,4}(\mathbb{S}^1 \times X, \{0\} \times X) &= KR^{0,5}(X) & \text{Class CII} \\
KR^{0,6}(\mathbb{S}^1 \times X, \{0\} \times X) &= KR^{0,7}(X) & \text{Class CI} \\
KR^{0,5}(\mathbb{S}^1 \times X, \{0\} \times X) &= KR^{0,6}(X) & \text{Class C}
\end{align*}
\]
Combining the loop and constant evolution components, two-gapped unitaries are classified according to the following table:

<table>
<thead>
<tr>
<th>( S )</th>
<th>( d = 0 )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<tbody>
<tr>
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<td>( \mathbb{Z} \times \mathbb{Z} )</td>
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<tr>
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<tr>
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<td>( \mathbb{Z}_2 \times \mathbb{Z}_2 )</td>
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<tr>
<td>D</td>
<td>( \mathbb{Z}_2 \times \mathbb{Z}_2 )</td>
<td>( \mathbb{Z}_2 \times \mathbb{Z}_2 )</td>
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<tr>
<td>DIII</td>
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<tr>
<td>AIII</td>
<td>( \mathbb{Z} \times \mathbb{Z} )</td>
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<tr>
<td>CII</td>
<td>( \emptyset )</td>
<td>( \mathbb{Z} \times \mathbb{Z} )</td>
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<tr>
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</tr>
<tr>
<td>CI</td>
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</tr>
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</table>
At an interface between two systems, the bulk topology may be manifested as edge modes.

If $n_C$ is the constant evolution invariant and $n_L$ is the unitary loop invariant, then the number of edge modes in each gap is:

$$n_{\pi} = n_L$$
$$n_0 = n_C + n_L$$
General Class D Hamiltonian

\[ H = \frac{i}{4} \sum_{ij} \gamma_i A_{ij} \gamma_j, \]

where $\gamma_i^\dagger = \gamma_i$ are Majorana fermions and $A_{ij}$ is an antisymmetric matrix.

In $d=1$, the static classification is $\mathbb{Z}_2$ [Kitaev 2001].
For a time dependent Hamiltonian, let

\[ O(t) = T \exp \left( \int_0^t A_{ij}(t') dt' \right). \]

- \( O(t) \) belongs to the special orthogonal group, \( SO(2n) \).
- \( H(0) \) and \( H(\pi) \) have the full symmetry of a 0d Class D Hamiltonian.

\[ \Rightarrow 1d \text{ Floquet cycles in Class D can be characterized by an invariant} \]
\[ \in \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ for } n > 2. \]

- This agrees with the classification obtained using K-theory.
Non-trivial Unitary Loop

- Consider a one-dimensional fermionic chain of length $K$ that has a two-state Hilbert space at each site (with annihilation operators $a_j$ and $b_j$):

  \[ a_1 a_2 a_3 a_4 a_5 a_6 \ldots a_{2K-1} a_{2K} \]
  \[ b_1 b_2 b_3 b_4 b_5 b_6 \ldots b_{2K-1} b_{2K} \]

  We define two sets of Majorana operators through

  \[
  \gamma^a_{2j-1} = a_j + a_j^\dagger, \quad \gamma^a_{2j} = \frac{a_j - a_j^\dagger}{i},
  \]
  \[
  \gamma^b_{2j-1} = b_j + b_j^\dagger, \quad \gamma^b_{2j} = \frac{b_j - b_j^\dagger}{i},
  \]

  which satisfy $\gamma^\dagger = \gamma$. 
Non-trivial Unitary Loop

Let

\[ H_{1a} = -\sum_j \left( a_j^+ a_j - \frac{1}{2} \right) = -\frac{i}{2} \sum_j (\gamma_{2j-1}^a \gamma_{2j}^a) \]
\[ H_{2a} = \frac{1}{2} \sum_j \left( -a_j^+ a_{j+1} - a_{j+1}^+ a_j + a_j a_{j+1} + a_{j+1}^+ a_j^+ \right) \]
\[ = \frac{i}{2} \sum_j (\gamma_{2j}^a \gamma_{2j+1}^a) , \]

with \( H_{1b}, H_{2b} \) defined identically in terms of the \( b \) operators and \( \gamma^b \) Majoranas.

\( H_{1a} \) and \( H_{2a} \) respectively correspond to the trivial and nontrivial phases of the 1D class D superconductor.
Evolve the system with 
\[ H_1 = H_{1a} + 2H_{1b} \text{ for } 0 \leq t \leq \pi \] and 
\[ H_2 = H_{2a} + 2H_{2b} \text{ for } \pi \leq t \leq 2\pi. \]

- \( H_{2a} \) and \( H_{2b} \) are topologically non-trivial while \( H_{1a}, H_{1b} \) are trivial.
- The evolution by \( H_{1a} \) pushes the Majorana mode of subsystem \( a \) to quasienergy \( \epsilon = \pi \),
- Evolving the closed system until \( t = 2\pi \) leads to a unitary that is the identity (up to an overall phase factor).
Unitaries for Open and Closed Systems

- Terms $\frac{i}{2} \gamma_a^a K \gamma_1^a$ from $H_{2a}$ and $\frac{i}{2} \gamma_a^b K \gamma_1^b$ from $H_{2b}$ are absent in the open system Hamiltonians.

- With

$$d = \frac{1}{2} (\gamma_a^a K + i \gamma_1^a), \quad d^\dagger = \frac{1}{2} (\gamma_a^a K - i \gamma_1^a),$$

$$e = \frac{1}{2} (\gamma_b^b K + i \gamma_1^b), \quad e^\dagger = \frac{1}{2} (\gamma_b^b K - i \gamma_1^b),$$

the unitary of the open system may be written

$$U_{\text{op}}(2\pi) = e^{\left[ \frac{\pi}{2} (\gamma_a^a K \gamma_1^a + 2 \gamma_b^b K \gamma_1^b) \right]} U_{\text{cl}}(2\pi)$$

$$= e^{\left[ -\frac{i\pi}{2} (d^\dagger d - dd^\dagger) - i\pi (e^\dagger e - ee^\dagger) \right]} U_{\text{cl}}(2\pi).$$
Effective Unitary for the Edge

- Effective Unitary for the edge:
  \[ U_{\text{eff}}(2\pi) = e^{-\frac{i\pi}{2}(d^\dagger d - dd^\dagger)} - i\pi(e^\dagger e - ee^\dagger)} \]

- Define effective parity operator \( \hat{P}_L \) at the left edge of the open chain

Then,

\[ \{ \hat{P}_L, U_{\text{eff}} \} = 0. \]
• Bulk topological invariant: $U_0(T)U_\pi^\dagger(T) = e^{i\nu\pi}\mathbb{I}$
  - correctly predicts $\pi$ Majorana mode
  - agrees with the non-interacting invariant

• Edge picture:
  - two fold degeneracy in spectrum of open system
  - has effective unitary of the form described above
Phase Transitions

- Suppose there is phase transition at $t_0$. i.e., no edge $\pi$ Majorana modes for $t < t_0$ but Majorana modes exist for $t > t_0$.
- At $t_0$, each bulk eigenstate $|\psi\rangle$ becomes part of a multiplet $\{ |\psi\rangle, d^\dagger |\psi\rangle, e^\dagger |\psi\rangle, d^\dagger e^\dagger |\psi\rangle \}$ after the transition.
- These form two pairs separated by quasienergy $\pi$ (modulo $2\pi$). The two states in each pair have opposite parity.
Bulk Boundary correspondence

- Suppose at some point beyond $t_0$, we reconnect the edges with an edge Hamiltonian

$$H' = h_d \left( d \dagger d - d \dagger d \right) + h_e \left( e e \dagger - e \dagger e \right),$$

and choose $h_d$ and $h_e$ so that the unitary at the end of the evolution is $I$.

- If we reconnect with a $\pi$ flux, $h_d$, $h_e$ change sign, the unitary goes to $-I$.

\[ \begin{align*}
\epsilon(t) & \quad \pi \\
\frac{\pi}{2} & \quad \frac{\pi}{2} \\
0 & \quad 0 \\
-\frac{\pi}{2} & \quad -\frac{\pi}{2} \\
-\pi & \quad -\pi \\
\end{align*} \]

\[ \begin{align*}
|\psi\rangle & \quad |\psi\rangle \\
|\psi\rangle & \quad |\psi\rangle \\
|\psi\rangle & \quad |\psi\rangle \\
|\psi\rangle & \quad |\psi\rangle \\
\end{align*} \]
Conclusions

- Topological Floquet systems may be classified using K theory.
- A gapped unitary evolution can be uniquely decomposed (up to homotopy) into a loop component and a constant evolution.
- The periodic table for free fermionic unitary evolutions may be obtained from the static periodic table through $G \rightarrow G \times G$.
- Unitary loops in one-dimensional interacting systems in class D may be classified by considering the effective edge unitary.
- A topological invariant is given by the phase change under a flux insertion.